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# The group-quark matrix ?

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## §1 The group-quark matrix ?

The main purpose of this article is to propose a problem. Let us consider the following $3 \times 3$ matrix whose entries are finite groups.

$$\mathcal{A} = \begin{bmatrix} U_4(2).2 & S_6(2) & O_8^+(2) \\ U_6(2).2 & Conway_2 & Conway_3 \\ {}^2E_6(2).2 & Fischer_4 & Monster \end{bmatrix}$$

The orders of relevant simple groups are :

$|U_4(2)| = 25920 = 2^6 3^4 5$
$|S_6(2)| = 1451520 = 2^9 3^4 5.7$
$|O_8^+(2)| = 174182400 = 2^{12} 3^5 5^2 7$
$|U_6(2)| = 2^{15} 3^6 5.7.11$
$|Conway_2| = 2^{18} 3^6 5^3 7.11.23$
$|Conway_3| = 2^{21} 3^9 5^4 7^2 11.13.23$
$|{}^2E_6(2)| = 2^{36} 3^9 5^2 7^2 11.13.17.19$
$|Fischer_4| = 2^{41} 3^{13} 5^6 7^2 11.13.17.19.23.31.47$
$|Monster| = 2^{46} 3^{20} 5^9 7^6 11^2 13^3 17.19.23.29.31.41.47.59.71$

Note that Conway groups are numbered according to their orders. In particular, $|Conway_1| = 2^{10} 3^7 5^3 7.11.23$. $U_4(2).2$ is the extension of $U_4(2)$ by an outer automorphism of order 2, and $U_6(2).2$ and ${}^2E_6(2).2$ are analogously defined.

The columns of the matrix $\mathcal{A}$ are indexed by the Dynkin diagrams of type $E_6, E_7$ and $E_8$. Appearing in the first row of $\mathcal{A}$ are the simple components of the Weyl groups of type $E_6, E_7$ and $E_8$. The correct indexing of the rows of the matrix $\mathcal{A}$ is left for the future research. We can, perhaps, index the rows of $\mathcal{A}$ by three generations of quarks *ud, cs, tb* (up-down,charm-strange,top-bottom).

Let us next give the 'transpose-inverse=tra-inv' of the matrix $\mathcal{A}$.

$$ {}^t\mathcal{A}^{-1} = \begin{bmatrix} 2^{1+4}.(S_3 \times S_3) & 2^{1+6^*}.(S_3 \times S_3) & 2^{1+8}.(S_3 \times S_3 \times S_3) \\ 2^{1+8}.U_4(2).2 & 2^{1+8}.S_6(2) & 2^{1+8}O_8^+(2) \\ 2^{1+20}.U_6(2).2 & 2^{1+22}Conway_2 & 2^{1+24}Conway_3 \end{bmatrix} $$

If $A_{ij}$ is the $(i,j)$ entry of the matrix $\mathcal{A}$, then the corresponding entry of ${}^t\mathcal{A}^{-1}$ is the cen<u>tra</u>lizer of an <u>inv</u>olution in the center of a Sylow 2-subgroup of the group $A_{ij}$. Here $2^{1+2n}$ denotes the extral-special group of order $2^{1+2n}$. An exception is $2^{1+6^*}$, which is almost extra-special but not exactly so. The main problem proposed here is : **Investigate the group-quark matrix $\mathcal{A}$ algebro-geometrically.**

## §2 $\Gamma_{27}$ and $\Gamma_{28}$

Let $S$ be the cubic surface defined in the projective space $P^4(\mathbb{C})$ by the equations :

$$ \begin{cases} x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0 \\ x_0 + x_1 + x_2 + x_3 + x_4 = 0. \end{cases} $$

The (projective) line defined by

$$ \begin{cases} x_0 = 0 \\ x_1 + x_2 = 0 \\ x_3 + x_4 = 0 \end{cases} $$

lies completley on the surface $S$. Applying the permutations on the index set $\{0,1,2,3,4\}$, 15 lines on $S$ can be obtained.

Next, let $\alpha$ $(= \frac{1\pm\sqrt{5}}{2})$ be a zero of the quadratic equation :

$$ X^2 - X - 1 = 0, $$

then the line defined by :

$$ \begin{cases} x_0 + \alpha x_3 + x_4 = 0 \\ x_1 + x_3 + \alpha x_4 = 0 \\ x_2 - \alpha(x_3 + x_4) = 0 \end{cases} $$

is also completely on the surface $S$. Applying the permutations on $\{0,1,2,3,4\}$ again, 12 lines can be obtained. Therefore there are altogether 27 lines on $S$. That this is the exact number of lines on $S$ comes from the theory of algebraic geomerty, although our special case itself was known already in the middle of the 19th century.

**Theorem.** A general (complex) cubic surface contains exactly 27 lines.

Let $\Gamma_{27}$ be the graph of 27 lines with their configuration on a general cubic surface. Then $\Gamma_{27}$ satisfies the following properties :

(1). Any line $A$ of $\Gamma_{27}$ meets exactly ten other lines of $\Gamma_{27}$. Those ten lines split into five pairs $(B_1, C_1), \ldots, (B_5, C_5)$, and if $i = 1, 2, 3, 4, 5$, then $B_i$ and $C_i$ meet and the triangle $AB_iC_i$ is formed. There are $5 \cdot 27/3 = 45$ triangles so formed. (Note. If $i \neq j$, then $B_i$ and $C_j$ do not meet. In particular, there are no three lines that meet at a point. This applies to a general cubic surface. A specialization of it may contain three lines that meet at a point.)

(2). Let $ABC, A'B'C'$ be any two triangles having no side in common. Then they determine uniquely a third triangle $A''B''C''$ such that each of three triples of lines $\{A, A', A''\}, \{B, B', B''\}, \{C, C', C''\}$ intersect and form three new triangles $AA'A'', BB'B'', CC'C''$.

Those two properties (1), (2) uniquely determines the configuration of 45 triangles formed by the elements of $\Gamma_{27}$.

**Theorem(C.Jordan).** $\mathrm{Aut}(\Gamma_{27}) \cong U_4(2).2 \cong \mathrm{Aut}(U_4(2))$

This is the $(1,1)$ entry of the matrix $\mathcal{A}$. The isomorphisms of simple groups

$$U_4(2) \cong S_4(3) \cong O_5(3) \cong O_6^-(2)$$

is significant in the history of group theory.

Let us next discuss the $(1,2)$ entry of the matrix $\mathcal{A}$. The graph of the quartic curve

$$x^4 + y^4 + x^2y^2 - 8(x^2 + y^2) + 16.25 = 0$$

is drawn at the end of this article.

It is easy to see that $28 = 4 + (12 \cdot 4/2)$ double tangents to the curve can be drawn. If the constant 16.25 is replaced by a number smaller than about 15.5 then four regions merge into a single region and if it is replaced by a number larger than about 17.5, then we get four convex regions and only 24 double tangents can actually be visible.

In general, it is known :

**Theorem.** A nonsingular (complex) plane curve of degree 4 possesses exactly 28 double tangents.

The number of double tangents to a nonsingular plane curve of degree $m$ is given by the formula of Plücker :

$$\text{Number of double tangents} = \frac{1}{2}m(m-2)(m^2-9).$$

Let $\Gamma_{28}$ be the set of 28 double tangents. The configuration satisfied by the 28 double tangents was investigated by Steiner, Aronhold and many others.

(1). (Steiner) Let $x_1, y_1$ be two distinct elements of $\Gamma_{28}$. Then there exist five pairs $(x_2, y_2), (x_3, y_3), \ldots, (x_6, y_6)$ of elements in $\Gamma_{28}$ and if we put

$$\mathfrak{S} = \{(x_i, y_i) | i = 1, 2, 3, \cdots, 6\}$$

then, the eight tangent points of any pair of double tangents $(x_i, y_i)$, $(x_j, y_j) \in \mathfrak{S}$ lie on a same conic (an irreducible plane curve of degree 2). $\mathfrak{S}$ is called a *Steiner complex.* $\Gamma_{28}$ possesses 63 Steiner complexes in total.

Let $P_1, \cdots, P_7$ be seven points given in the complex plane. The cubic curves passing through these seven points form a vector space $\mathfrak{T}$. Every pair of curves $\{C_1, C_2\}$ of $\mathfrak{T}$ intersect two more points by Bézout's theorem. If these two points coinside then the pair $\{C_1, C_2\}$ possesses a common tangent. The totality of common tangents so obtained forms a plane curve $D'$ of class 4, or equivalently the dual curve of a plane curve of degree 4.

The dual of the statement above will read as follows.

(2)(Aronhold). Let $L_1, \cdots, L_7$ be seven lines on the plane. The totality of all curves of class 3 containing these seven lines forms a vector space $\mathfrak{T}'$. Every pair of curves $\{C_1', C_2'\}$ in $\mathfrak{T}'$ contains two more lines $\{L_8, L_9\}$ in common. If $L_8 = L_9$, then the pair $\{C_1', C_2'\}$ possesses a tangent point $z$ and $z$ is on a curve $D$ of degree 4 uniquely determined by $L_1, \cdots, L_7$. Moreover, $L_1, \cdots, L_7$ are double tangents of this curve $D$.

Let $D$ be the curve of degree 4 uniquely determined by the seven lines $\{L_1, \cdots, L_7\}$. Then $D$ possesses 28 double tangents $\Gamma_{28} = \{L_1, L_2, \ldots, L_{28}\}$. Moreover, the following properties hold.
(i). $L_1, \cdots, L_7$ is a maximal asyzygetic set (defined below) of $\Gamma_{28}$.
(ii). The remaining 21 double tangents are rationally constructible by $L_1, \cdots, L_7$ (their coefficients are rational functions of the coefficients of $L_1, \cdots, L_7$ ).
(iii). Every curve of degree 4 without double points can be obtaind by this construction.
(iv). Every asyzygetic set of seven double tangents of $\Gamma_{28}$ defines $D$.

Let $L_1, L_2, L_3$ be three distinct lines in $\Gamma_{28}$. Those three lines determine six tangent points. If those six tangent points are on a same conic, then the triple $\{L_1, L_2, L_3\}$ is called *syzygetic.* In the contrary case, the triple is called *asyzygetic.* A subset $S$ of $\Gamma_{28}$ is called asyzygetic if every triple of $S$ is asyzygetic.

Let us call a maximal asyzygetic seven-line set mentioned in (i) an *Aronhold set.* Therefore, an Aronhold set is a maximal asyzygetic subset of $\Gamma_{28}$ consisiting of seven elements. It is known that $\Gamma_{28}$ contains exactly 288 Aronhold sets.

**Theorem(Jordan).** $Aut(\Gamma_{28}) \cong S_6(2)$.

Note that $|S_6(2)| = 288 \times 7!$. In fact, $S_6(2)$ transitively permutes all Aronhold sets and the fixing subgroup of an Aronhold set $A$ acts as the symmetric group of degree 7 on $A$.

$\Gamma_{28}$ can not be determind only by vertices and edges since $\mathrm{Aut}(\Gamma_{28})$ acts doubly transitively on the 28 points. Therefore, $\Gamma_{28}$ is not a graph in an usual sense.

Let $L_1, L_2$ be a pair of elements in $\Gamma_{28}$, then there are 10 elements $X$ in $\Gamma_{28}$ such that $\{L_1, L_2, X\}$ is a syzygetic triple. In fact, all such $X$ are in the Steiner complex determined by the pair $\{L_1, L_2\}$. Therefore, $\Gamma_{28}$ possesses $28.27.10/6 = .1260$ syzygetic triples. If all syzygetic triples are given in $\Gamma_{28}$, then the configuration of $\Gamma_{28}$ is completely determined. The author is not aware if any combinatorial characterization of $\Gamma_{28}$ is known. (Note. A combinatorial characterization of $\Gamma_{27}$ is known as mentioned in this article before.)

Let $L$ be an element of $\Gamma_{28}$. Consider $\Gamma'_{27} = \Gamma_{28} \backslash \{L\}$. For a pair of elements $X, Y$ in $\Gamma'_{27}$, if $L, X, Y$ is syzygetic, connect $X$ and $Y$ by an edge. Then a graph of 27 vertices and 135 edges is obtained. The $\Gamma'_{27}$ is isomorphic with $\Gamma_{27}$ discussed before (Geiger, 1869).

We have thus obatined the (1,2) entry of the matrix $\mathcal{A}$.

**Problem**. Define the (1,3) entry of the group-quark matrix $\mathcal{A}$ algebro-geometrically.

Since
$$[O_8^+(2) : S_6(2)] = 120,$$
the algebro-geometric model on which $O_8^+(2)$ acts should contain 120 elements in it. Let us denote the object by $\Gamma_{120}$. The fixing subgroup of a point $\alpha$ of $\Gamma_{120}$ should be $S_6(2)$.

Therefore, $\Gamma_{120}$ is, as an $O_8^+(2)$-set, equivalent to the quotient space $O_8^+(2)/S_6(2)$. The action of $O_8^+(2)$ on $O_8^+(2)/S_6(2)$ is well known and it induces a rank 3-permutation representation. Equivalently one point stabilizer $S_6(2)$ has exactly two orbits on the remaining 119 points $\Gamma_{120} \backslash \{\alpha\}$. The suborbit lengths are 56 and 63, and the stabilizer of a point in $S_6(2)$ is $U_4(2)$ or $E_{32}.S_5$ respectively. Let us write
$$\Gamma_{120} = \{\alpha\} + \Delta + \Omega$$
where, $|\Delta| = 56, |\Omega| = 63$.

We are assuming that the configuration graph of $\Gamma_{120}$ contains $\Gamma_{28}$ as a subgraph. Therefore, we should be able to identify $\Delta$ and $\Omega$ in terms of $\Gamma_{28}$. $\Omega$ is of length 63 and so it is natural to assume that $\Omega$ is the totality of all Steiner complexes.

There are 28 double tangents and so obviously there are 56 tangent points. Therefore, it is natural again to choose $\Delta$ to be the set of all (double) tangent points of the plane curve of degree 4 that we initially began with.

In Heinrich Weber's Lehrbuch der Algebra, Vol II(1899), there is a 50 page chapter entirely devoted to the structure of $\Gamma_{28}$. In it, it is proved also that $S_6(2)$ is the automorphism group of the configuration.

There are other 120 mathematical objects.
(1). A nonsingular plane curve of degree 5 posseses 120 double tangents (easy by Plücker's formula).
(2). There is a curve ( called del Pizzo surface) of degree 6 and of genus 4 possessing 120 tritangents planes.
(3). The root system of type $E_8$ possesses 240 roots. If the sign of each root is ignored then a set $\Gamma$ of 120 objects and its graph are obtained.

It must be an interesting problem to investigate the configuration $\Gamma_{120}$ purely group theoretically also.

## §3 The second and third rows of $\mathcal{A}$.

The second and third rows of the group-quark matrix are up in the air at this moment. McKay [Finite Groups, Proceedings of Symposia in Pure Mathematics, Vol. 37, Amer. Math. Soc. 1980] observed that if $s$ and $t$ are involutions of the Monster both of which are conjugate to the involutions of $2A$ type, then its product $st$ belongs to the conjugacy classes of the Monster of type

$$1A, 2A, 3A, 4A, 5A, 6A, 3C, 4B, 2B.$$

Recall that if $-\alpha_0$ is the highest root of the Lie algebra of type $E_8$, then

$$1\alpha_0 + 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 5\alpha_4 + 6\alpha_5 + 3\alpha_6 + 4\alpha_7 + 2\alpha_8 = 0.$$

The numbers $\{1, 2, 3, 4, 5, 6, 3, 4, 2\}$ are called the weights of $E_8$. McKay lists $Fischer_3$ and $Fischer_4$ as groups having similar property with respect to $E_6$ and $E_7$, respectively. $Fischer_3$ is replaced by ${}^2E_6(2)$ in this article, since it fits better if we consider the (2,1) entry of the tra-inv ${}^t\mathcal{A}^{-1}$ of the group-quark matrix.

Similar coincidences between weights of Dynkin diagrams and orders of groups elements have been observed by Glauberman and Norton [to appear in the Proceedings of Monster Workshop at Montreal, 1999] . At Kyoto symposium, the (2,1) and (3,1) entries of the matrix $\mathcal{A}$ were the sporadic simple groups *Suzuki* and $Fischer_3$, respectively. The new entries $U_6(2)$ and ${}^2E_6(2)$, however, appear to fit its tra-inv matrix ${}^t\mathcal{A}^{-1}$ better, although leaving the main realm of the 3-transposition groups may be a problem.

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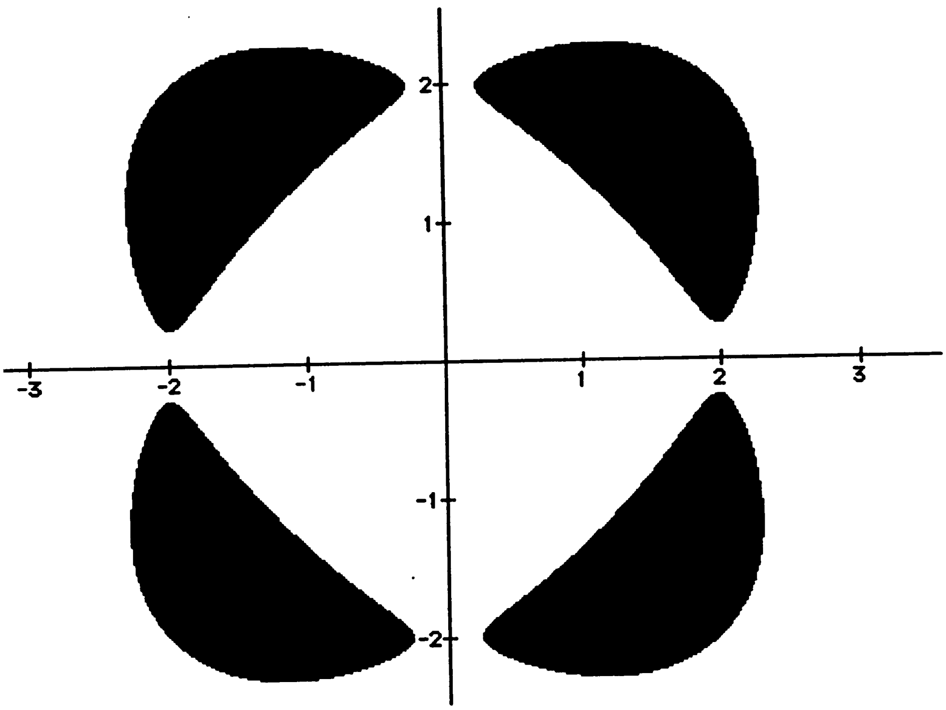


$$x^4 + y^4 + x^2y^2 - 8(x^2 + y^2) + 16.25 < 0$$